Trigonometric, Hyperbolic, and Inverse Functions

Continuing the Logarithm, and the Circular/Hyperbolic Functions

Lecture Notes

October 6, 2026

1 The Logarithm (continued)

Theorem

If $\ell(z)$, a branch of log, is defined on $B(z_0, r)$, then $\ell(z)$ is analytic on $B(z_0, r)$ and $\ell'(z) = 1/z$.

Proof. Analyticity is a homework problem. Given analyticity, differentiate $e^{\ell(z)} = z$: by the chain rule and the inverse function rule,

$$\ell'(z) = \frac{1}{(e^w)'\big|_{w=\ell(z)}} = \frac{1}{e^{\ell(z)}} = \frac{1}{z}. \qquad \blacksquare$$

Theorem

On $\mathbb{D} = B(0,1)$,

$$\operatorname{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n.$$

Proof. Let $\ell(z) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$. Then

$$\ell'(z) = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z} = \big(\operatorname{Log}(1+z)\big)'.$$

So $\ell(z) - \operatorname{Log}(1+z) \equiv \text{const} = \ell(0) - \operatorname{Log} 1 = 0$. $\square$

2 Trigonometric Functions

Returning to Euler's equations, $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$ give

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Definition (sine and cosine)

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z := \frac{e^{iz} - e^{-iz}}{2i}.$$

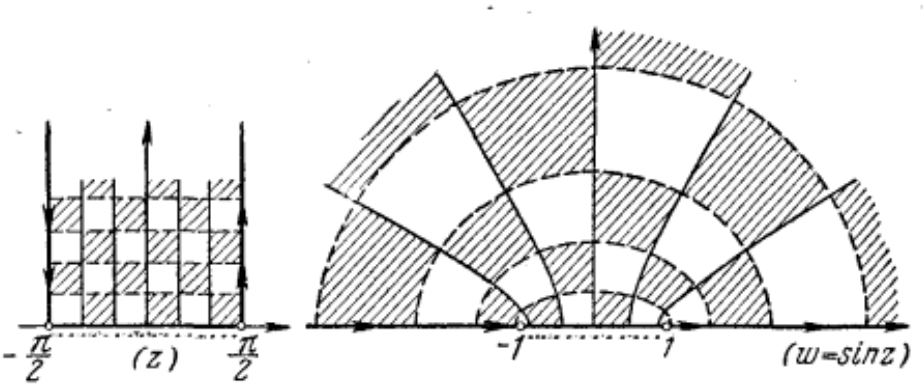


Figure 1: The map $w = \sin z$. Since $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$, the half-strip $\{|\operatorname{Re} z| < \pi/2,\ \operatorname{Im} z > 0\}$ is mapped onto the upper half-plane: horizontal segments $\{\operatorname{Im} z = c\}$ go to half-ellipses with foci ± 1, and vertical rays $\{\operatorname{Re} z = c\}$ to branches of confocal hyperbolas.

Same power series as in the real case:

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \qquad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!},$$

both 2π-periodic.

Basic identity.

$$\cos^2 z + \sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 = 1.$$

Derivatives.

$$\cos' z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)' = i\,\frac{e^{iz}-e^{-iz}}{2} = -\sin z, \qquad \sin' z = \cos z.$$

Tangent.

$$\tan z := \frac{\sin z}{\cos z} = -i\,\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}, \qquad \pi\text{-periodic:}$$

$$\tan(z+\pi) = -i\,\frac{e^{iz+i\pi}-e^{-iz-i\pi}}{e^{iz+i\pi}+e^{-iz-i\pi}} = -i\,\frac{(-1)(e^{iz}-e^{-iz})}{(-1)(e^{iz}+e^{-iz})} = \tan z.$$

Its derivative is

$$\tan' z = \frac{4}{(e^{iz}+e^{-iz})^2} = \frac{1}{\cos^2 z}.$$

3 Hyperbolic Functions

Definition (hyperbolic sine and cosine)

$$\cosh z := \frac{e^z+e^{-z}}{2} = \cos(iz), \qquad \sinh z := \frac{e^z-e^{-z}}{2} = -i\sin(iz).$$

These satisfy $\cosh^2 z - \sinh^2 z = 1$, $\cosh' = \sinh$, $\sinh' = \cosh$, and

$$\coth z := \frac{\cosh z}{\sinh z}.$$

The pair $(\cosh t, \sinh t)$ parametrizes (one branch of) the hyperbola $x^2 - y^2 = 1$.

4 Inverse Functions

Arctangent (multivalued!). Solve $z = \tan w$:

$$z = -i\,\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \implies e^{2iw} = \frac{1+iz}{1-iz} = \frac{i-z}{i+z} \implies \boxed{w = -\frac{i}{2}\log\!\left(\frac{i-z}{i+z}\right)}, \qquad z \neq \pm i.$$

As $z \to \infty$: $-\dfrac{i}{2}\log(-1) = -\dfrac{i}{2}(\pi i + 2\pi i k) = \dfrac{\pi}{2} + \pi k$, matching the real arctangent's branch values at infinity.

Inverse hyperbolic sine. The same method gives

$$\operatorname{arcsinh} z = \log\!\left(z + \sqrt{z^2+1}\right).$$